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Correlated percolation and the correlated resistor network

Paul J M Bastiaansen† and Hubert J F Knops

Institute for Theoretical Physics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands

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Abstract. We present some exact results on percolation properties of the Ising model, when the range of the percolating bonds is larger than the nearest neighbours. We show that the phase diagram for next-nearest neighbour percolation can be exactly obtained from the nearest-neighbour case, which implies that the percolation threshold, T_p , is still equal to the Ising critical temperature T_c . In addition, we present Monte Carlo calculations of the finite size behaviour of the correlated resistor network defined on the Ising model. The thermal exponent, t , of the conductivity that follows from it is found to be $t = 0.2000 \pm 0.0007$. We observe no corrections to scaling in its finite size behaviour.

1. Introduction

The connection between percolation and the Ising model has been a popular subject for a long time. One considers so-called Ising clusters made up of nearest-neighbour spins with the same spin value. The connectivity behaviour of these clusters is called correlated site percolation, as the probability distribution of the percolating and non-percolating sites is a correlated one.

The interest in this problem arose because these Ising clusters were believed to have the same properties as the droplets in the droplet model [1], i.e. they should diverge at the Ising critical point with the same critical exponents as those of the Ising model. It became clear that they did indeed diverge [2] at the Ising critical point, but not [3] with Ising exponents. An alternative cluster definition was needed to have clusters with the properties of droplets in the droplet model. These new clusters are precisely the random clusters from the random cluster formulation of the Potts model, which work is due to Fortuin and Kasteleyn [4]. In the context of the Ising model, these clusters are called Coniglio–Klein clusters [5], and are defined by putting bonds between each pair of nearest-neighbour up-spins, but now with a probability $p = 1 - \exp(-2K)$, where K is the Ising coupling. Not all bonds of the Ising clusters appear in the Coniglio–Klein clusters, such that the latter are, in that sense, ‘smaller’ than Ising clusters. The Coniglio–Klein clusters display [5] the correct critical behaviour: they diverge at the Ising critical point, their linear size diverges as the Ising correlation length, and the mean cluster size behaves as the susceptibility.

Both the Ising clusters and the Coniglio–Klein clusters have their percolation point at the Ising critical temperature, albeit with different critical behaviour. The full picture of this cluster behaviour emerged [6, 7] when the behaviour of both types of clusters was identified with the phase diagram of the q -state dilute Potts model in the limit $q \rightarrow 1$. The tricritical

† E-mail address: paulb@tvs.kun.nl

point in this phase diagram describes the behaviour of the Ising clusters. This tricritical point falls in the same universality class as the Ising critical point [8] in the sense that the central charge is $c = \frac{1}{2}$, but the critical exponents involved in the behaviour of the Ising clusters do not fit into the Kač-table [9]; they correspond to half-integer values of the unitary grid. The Coniglio–Klein clusters are described by the $1 + 1$ state symmetric fixed point in this phase diagram.

Our motivation to reconsider the problem of correlated site percolation is not the droplet theory of the Ising model but arises from a study of correlated resistor networks; see below. In this paper, we consider Ising clusters that are made up of bonds with a larger percolation range, that is, bonds are placed between nearest-neighbour up-spins, but also between next- and further-neighbouring pairs of equal spins. Let us, throughout this paper, denote the clusters consisting of bonds between nearest and next-nearest neighbour spin pairs by nnn-clusters. In this language the Ising clusters are nn-clusters.

It is immediately clear that, if one considers bonds with a longer and longer percolation range, clusters get bigger and the percolation threshold eventually will move to a temperature, T_p , that is lower than T_c , the Ising critical temperature. In that case, the type of correlation is expected to be random percolation. In the limit of percolating bonds with an infinite range, the percolation temperature moves to $T_p = 0$ and there is crossover to classical critical behaviour [10]. In three dimensions this effect of a shift in the percolation threshold has previously occurred with Ising clusters (nn-clusters): the percolation threshold lies at a temperature a few percent below T_c [11], whereas the Coniglio–Klein clusters have their percolation threshold at T_c . In two dimensions it is known, as stated above, for nn-clusters T_p coincides with T_c . It was believed [12] that for nnn-clusters $T_p < T_c$, however, in this paper we shall show that $T_p = T_c$ for nnn-clusters.

These alternative cluster definitions can be useful in some applications of correlated percolation. In another paper [13], we present a model, based on correlated percolation, to explain the experimental results for colossal magnetoresistance. The latter phenomenon is presently a hot topic in solid state physics [14]. Our model is a correlated resistor network, obtained by replacing bonds with resistances yielding an effective resistance as an Ising expectation value. In the present work, we present the technical analysis of the correlated percolation model with percolating bonds having a longer percolation range than nearest neighbour. In particular, the resulting phase diagram is used for understanding the experimental results of colossal magnetoresistance. The correlated resistor network has, to our knowledge, never been studied in any literature. We performed Monte Carlo calculations to measure the critical exponents of the CRN. These calculations are also presented in this paper.

2. The model

We will be concerned with the usual Ising model on a square lattice with Hamiltonian

$$H = -K \sum_{\langle ij \rangle} S_i S_j - h \sum_j S_j$$

where K is the inverse temperature and h is the magnetic field. The first summation is over nearest neighbours only. Note that, throughout this paper we will be considering two different ranges of percolating bonds, i.e. nearest and next-nearest neighbour. Note, however, that the Ising *interaction* is, throughout the paper, exclusively via nearest-neighbour couplings, as in the above Hamiltonian.

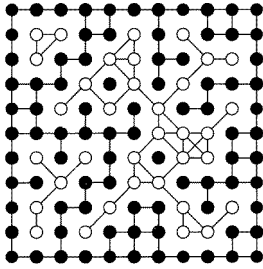


Figure 1. An illustration of the theorem presented in the text. On the sites of a lattice, black and white spins are placed. Spins on the boundary are always black. Bonds between black spins are present between each pair of black spins that are nearest neighbours. The same is true for the white spins, but here bonds are drawn if the spins are next-nearest neighbours as well. It can easily be inferred that a face of a black cluster is either empty or wholly occupied with one and only one white cluster. Notice that the inclusion of next-nearest neighbour bonds is essential.

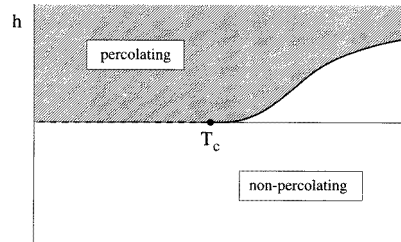


Figure 2. The phase diagram for percolation of Ising clusters. Up-spins are considered percolating, and clusters are defined by putting bonds between each neighbouring pair of up-spins. T_c is the Ising critical point, and T and h are temperature and magnetic field, respectively. The heavy full curve is a critical line of percolation that is in the universality class of random percolation. T_c is a tricritical point for percolation, and the broken line is a first-order transition, for Ising as well as for percolation.

First consider the nn-clusters by putting bonds between all neighbouring pairs of up-spins, such as the clusters made of the black spins in figure 1. (To adopt the terminology of the figure, we will label the Ising spins as black and white instead of up and down.) In the same figure, also the nnn-clusters are illustrated but now for the white spins. Here bonds are put between next-nearest neighbouring pairs of white spins as well.

The percolation phase diagram for the nn-clusters is known [7] and shown in figure 2. The heavy full curve is a critical percolation transition. Its exact location is not known, but the location of the endpoint at $T = T_c$ is exact [2]. The other endpoint, at $T = \infty$, corresponds to random percolation and lies at the value of h corresponding to the percolation threshold [15] $p_c \approx 0.5927$ for random percolation. This value denotes the density of up-spins. The corresponding value of h is $h \approx 0.188$. The critical percolation line is in the universality class of random percolation, described by the critical $q = 1$ state Potts model. The line merges smoothly with the T -axis at the Ising critical point. For percolation, this point turns out [7] to be a tricritical point; it is the tricritical point of the $q = 1$ state dilute Potts model, where, apart from the usual Potts spins, also vacancies are allowed.

From this phase diagram the corresponding diagram for nnn-clusters can be derived. Figure 1 illustrates the theorem [16] that will be needed for this derivation of the phase diagram. It can easily be seen from this figure that there exists a geometrical relation between the black nn-clusters and the white nnn-clusters. The theorem states that *every face of a black nn-cluster is either empty or is wholly occupied by one and only one white nnn-cluster*. A face is a closed area surrounded by elementary loops consisting of the bonds of one cluster. Let us introduce the following notation: let f_B , c_B , b_B and n_B be the number of faces, clusters, bonds and sites respectively that correspond to the black spins and nn-clusters in a certain spin configuration. For the white spins and nnn-clusters, these quantities are f_W^* , c_W^* , b_W^* and n_W^* respectively, where the star denotes the fact that it concerns nnn-clusters. The theorem then states that

$$f_B = f_B^{(0)} + c_W^*$$

where $f_B^{(0)}$ denotes the number of empty black faces. The number of faces, bonds, sites and clusters is furthermore coupled via Euler's equation

$$f_B = b_B - n_B + c_B + 1.$$

This relation is easily derived by induction and holds for non-cyclic boundary conditions, e.g. all spins on the boundary are black. Combining these two relations yields

$$c_W^* = c_B + b_B - n_B - f_B^{(0)} + 1.$$

Thus, this relation expresses the number of white nnn-clusters, c_W^* , in terms of the number of black nn-clusters c_B . Apart from the numbers of clusters it involves only locally defined quantities: bonds, sites and empty faces.

The above relations are purely geometrical and hence are completely independent of the probability distribution of the black and white spins. Therefore they are valid as Ising expectation values as well:

$$\langle c_W^* \rangle = \langle c_B \rangle + \langle b_B \rangle - \langle n_B \rangle - \langle f_B^{(0)} \rangle + 1 \quad (1)$$

where the brackets denote the expectation values. Note that the numbers of bonds, sites and empty black faces are simply local Ising operators, that is

$$\langle b_B \rangle = \frac{1}{4} \sum_{\langle ij \rangle} \langle (1 + S_i)(1 + S_j) \rangle$$

which amounts to counting the number of black bonds. In the same way

$$\langle n_B \rangle = \frac{1}{2} \sum_j \langle 1 + S_j \rangle$$

counts the number of black spins and

$$\langle f_B^{(0)} \rangle = \frac{1}{16} \sum_{\langle ijkl \rangle} \langle (1 + S_i)(1 + S_j)(1 + S_k)(1 + S_l) \rangle$$

counts the number of empty, black faces, where $\langle ijkl \rangle$ denotes a summation over each elementary plaquette of the square lattice.

The expectation value of the number of clusters plays the role of the free energy in a percolation problem [16]. It becomes non-analytic at a percolation transition. From equation (1) we see that c_W^* can only become critical when c_B is critical, that is, when the black spins are at their percolation threshold, or when the Ising expectation values become non-analytic, that is, at the Ising critical point and at the coexistence line $T < T_c$, $h = 0$. This immediately yields the phase diagram for percolation of the black nnn-clusters: it is the mirror image of that of the nn-clusters with h replaced by $-h$. This phase diagram is shown in figure 3.

It is somewhat surprising that extending the range of percolating bonds to next-nearest neighbours does not alter the percolation threshold at the T -axis. So for next-nearest neighbour percolating bonds $T_p = T_c$ still holds. It was expected [12] that a larger percolation range immediately causes a lower percolation threshold $T_p < T_c$. Due to the above geometrical relations this is not the case. Upon extending the percolation range even further than the next-nearest neighbour, however, no such relations exist, and we expect the percolation threshold, T_p , indeed to drop below the Ising critical temperature T_c .

In the latter case, the topology of the phase diagram will change. There is no tricritical point any more, and the critical percolation line will end somewhere at the T -axis below T_c . Beyond this point, there is still a first-order transition for percolation. We expect the percolation point, T_p , in that case to be a critical endpoint. Such a point is expected [17] to have, in addition to the critical exponents of the universality class of the critical line, also

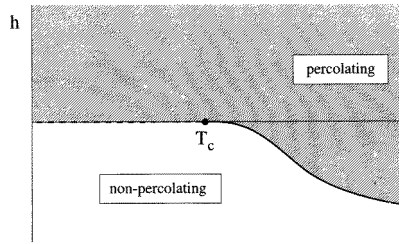


Figure 3. The same phase diagram as in figure 2, but now for nnn-clusters. Percolating bonds are drawn between pairs of up-spins that are nearest or next-nearest neighbours. The phase diagram is the same as that for nn-clusters but with h replaced with $-h$.

a first-order exponent $y = 2$. Indeed, when the range of percolating bonds becomes very large, it will eventually become larger than the Ising correlation length. In such a case, correlations in the probability distribution of empty and occupied sites are only present on a much smaller scale than the range of percolating bonds. This strongly suggests that the universality class of the endpoint of the critical line is that of random percolation.

Eventually, when the percolation range approaches infinity, percolation is believed to exhibit classical critical behaviour [10], that is, displays mean-field exponents.

3. Scaling analysis

From universality, we expect both types of critical behaviour, nn-percolation and nnn-percolation, to be in the same universality class. In the light of equation (1), this statement is less obvious than it seems. The singular behaviour of the ‘free energy’ c_W^* of nnn-clusters is expressed in c_B but also in Ising operators, and both c_B and the Ising expectation values become critical. Hence, in addition to the critical behaviour of c_B also Ising exponents show up. In particular, when c_W^* is considered as a function of the scaling field u_1 , in addition to the exponent $2/y$ that is expected to describe the non-analytic behaviour of the free energy c_B , also the magnetization exponent $\frac{1}{8}$ is present.

The following analysis relies on the work of Stella and Vanderzande [7] on correlated percolation in the Ising model, which, in its turn, goes back to results from several authors on the q -state Potts model that are reviewed by Nienhuis in [8]. Their work yields the exact critical behaviour of the q -state (dilute) Potts model. We will only globally repeat the analysis, and refer the interested but unfamiliar reader to these references.

Nn-percolation is in the universality class of the dilute q -state Potts model in the limit of $q \rightarrow 1$. Its Hamiltonian [6, 7] is

$$\mathcal{H} = -L \sum_{\langle ij \rangle} n_i n_j - \Delta \sum_j n_j - J \sum_{\langle ij \rangle} n_i n_j (\delta_{\sigma_i, \sigma_j} - 1) - H \sum_j n_j (\delta_{\sigma_j, 1} - 1). \quad (2)$$

The variables $n_i = 0, 1$ are the Potts lattice gas variables. Potts spins $\sigma_i = 0, \dots, q$ are present on the sites where $n_i = 1$.

For $q = 1$ the Hamiltonian becomes equal to the Ising Hamiltonian and completely independent of J and H . Substituting $S_i = 2n_i - 1$ turns the lattice gas variables into Ising variables. The Hamiltonian then becomes, apart from a constant,

$$\mathcal{H} = -K \sum_{\langle ij \rangle} S_i S_j - h \sum_j S_j$$

with $K = L/4$ and $h = 2L + \Delta/2$. This means that the free energy resulting from equation (2) for $q = 1$ is equal to that of the Ising model. The full free energy is

$$f(L, \Delta, J, H, q) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z^{(N)}(L, \Delta, J, H, q)$$

where N denotes the number of sites on the lattice. The generating function [4] for percolation is

$$c(L, \Delta, J, H) = \left. \frac{df(L, \Delta, J, H, q)}{dq} \right|_{q=1}. \quad (3)$$

Quantity c is, with $H = 0$, the expectation value of the number of clusters. It plays the role of the free energy in percolation problems, but is actually the derivative of a free energy with respect to a symmetry parameter. The limit $J \rightarrow \infty$ corresponds to nn-clusters, and $J = 2K$ yields the Coniglio–Klein clusters.

In the language of the renormalization group, the tricritical point in the $q = 1$ dilute Potts model has four relevant exponents and corresponding scaling fields, two thermal and two magnetic ones. Expressing the free energy in terms of these scalings fields u_i , the scaling relation is

$$f(u_1, \dots, u_4, q) = b^{-2} f(b^{y_1} u_1, \dots, b^{y_4} u_4, q) \quad (4)$$

where b is the renormalization length, and the tricritical point is located at $u_1 = \dots = u_4 = 0$. The exact values of the exponents are [8, 18] $y_1 = \frac{15}{8}$ and $y_2 = 1$ for the thermal exponents (which are just the Ising critical exponents) and $y_3 = \frac{187}{96}$ and $y_4 = \frac{115}{96}$ for the magnetic exponents. Note that the ‘field’ q , which is a symmetry parameter, cannot change under renormalization. Differentiating this free energy with respect to q yields the percolation free energy. Applying this to the above scaling relation yields, apart from the direct derivative to q , also derivatives with respect to the thermal scaling fields u_1 and u_2 . These fields are nonlinear functions of the coupling constants and vanish at the critical point, so correct to first order in the couplings, they are

$$u_1 = K - K_c(q) + \dots \quad (5)$$

$$u_2 = h - h_c(q) + \dots \quad (6)$$

when $J = \infty$. The location of the tricritical point thus depends on the value of q . The remaining two fields u_3 and u_4 are magnetic scaling fields; they correspond to Potts-like magnetic fields. The Potts-model can only be critical if those fields are zero, regardless of the value of q , which means that the derivative of these fields with respect to q yields zero.

From these remarks it follows that in the expression for c several derivatives are present: a direct derivative with respect to q and derivatives with respect to the thermal scaling fields:

$$c = \left. \frac{\partial f}{\partial q} + \frac{\partial f}{\partial u_1} \frac{\partial u_1}{\partial q} + \frac{\partial f}{\partial u_2} \frac{\partial u_2}{\partial q} \right|_{q=1}. \quad (7)$$

Applying equation (4) to this expression and taking the limit of $q \rightarrow 1$ yields the critical behaviour of c . Note that the last two terms of the right-hand side of (7) are, with $q = 1$, just derivatives of the Ising free energy: they give the Ising energy and magnetization, and thus yield Ising critical behaviour. The values of the exponents y_1 and y_2 are the Ising values $y_1 = 1$ and $y_2 = \frac{15}{8}$. This gives for the singular behaviour of c as a function of the thermal field u_1

$$c \sim A_1 |u_1|^{2/y_1} + A_2 |u_1|^{(2-y_1)/y_1} + A_3 |u_1|^{(2-y_2)/y_1}. \quad (8)$$

The most relevant of these exponents is $(2 - y_2)/y_1 = \frac{1}{8}$, the exponent of the Ising order parameter. The singular behaviour of c thus is described by an exponent $\frac{1}{8}$, in contrast to what is expected for the behaviour of a free energy-like quantity: normally in a free energy only the first term in equations (7) and (8) appears, yielding a critical exponent $2/y_1 = 2$ for the percolation free energy c . In our case, this exponent indeed appears in the expression for c , but is dominated by the exponent $\frac{1}{8}$. The fact that an exponent less than 1 appears in the ‘free energy’ (which is forbidden by stability) is due to the fact that c is not a true free energy but a derivative of a free energy with respect to the symmetry parameter q , see equation (3).

The order parameter P (the density of sites in the infinite cluster) and the ‘susceptibility’ (the mean cluster size) arise from differentiating c with respect to H . The free energy, f , itself becomes completely independent of H in the limit for $q \rightarrow 1$, which means that only the derivatives of $\partial f/\partial q$ with respect to H yield a non-zero result. The other terms in equation (7) vanish upon taking the derivative. This means that the critical behaviour of the order parameter and the susceptibility is not affected by the ‘wrong’ critical behaviour of c .

From the scaling results (8) of c we can derive the critical behaviour of the nnn-clusters using equation (1). The dominating exponent in the right-hand side of (1) is again $\frac{1}{8}$, which appears in c but also in the Ising magnetization $\langle n_B \rangle$. This shows that the critical behaviour of c and c^* , that is, of nn-clusters and nnn-clusters, is governed by the same exponents.

The same must be true for the order parameter, P , and the susceptibility of nnn-percolation. Again these quantities arise from differentiating the free energy c^* with respect to the Potts field H . Although c^* is different from c in its dependence on H , both amount to a dilute $q = 1$ states Potts model. With the identification of the critical behaviour of c and c^* it follows that at T_c both free energies must be at a tricritical point in the full phase diagram of this model.

We conclude that percolation of nn-clusters and of nnn-clusters is in the same universality class. The critical behaviour of the percolation quantities is governed by the same set of critical exponents.

4. The correlated resistor network

The behaviour of the Ising model as a resistor network is relevant for our work on colossal magnetoresistance [13]. A percolation model is turned into a resistor network [19] by replacing the bonds with resistors. Non-percolating bonds are left empty (that is, have infinite resistance), bonds that are present get a unit resistance. This can be done both for nn-clusters and nnn-clusters. The assignment of resistors is depicted in figure 4.

For random percolation the resistor network is a random resistor network. The corresponding correlated resistor network has, as far as we know, never been studied. In this section we present our calculations on the correlated resistor network. The interest in resistor network problems is in the expectation value of the overall resistance of the (infinite) lattice. To be more precise: consider a lattice consisting of $L \times L$ spins, where L eventually is sent to infinity, and keep the lower row at a fixed potential $V = 0$ and the upper row at $V = 1$. The interest is in the overall conductance, σ , of the lattice, which is in this case equal to the expectation value of the current.

The phase diagram of the resistor network is of course the same as that of its percolation counterpart: when there is percolation, the conductance is finite, and the conductance is zero when there is no percolation. Experimental results for colossal magnetoresistance show a

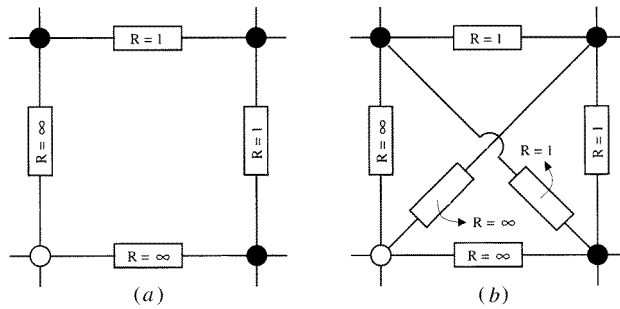


Figure 4. The assignment of resistors to a spin configuration. Assignment for nn-clusters is shown in (a), for nnn-clusters in (b). Here black spins (spin-up) are considered percolating, so bonds between two black spins have a unit resistance. Bonds between black and white spins or between two white spins get an infinite resistance.

sharp increase in the resistance as a function of temperature at or nearby the Curie point T_c . The resistance drops sharply both above and below this point. From our phase diagrams it is directly clear that the diagram of figure 2 is ruled out. So conduction via next-nearest neighbour bonds should at least be present to produce the correct phase diagram.

Turning, however, to critical exponents the situation is different. In this case it is the exponent t governing the vanishing conductance, σ , upon approaching the percolation threshold:

$$\sigma(T) \sim |T - T_p|^t \quad \text{for } T \rightarrow T_p.$$

Based on universality (again confirmed for the percolation exponents) one may well assume that the exponent t is the same for nn- and nnn-networks. Hence we studied the (simpler) case of the resistor network with only nn-clusters.

The random resistor network is a notoriously unsolved problem in statistical physics, but there are good numerical estimates of the exponent t . The best estimate [20] in two dimensions known to us is $t = 1.299 \pm 0.002$. To obtain the value of t for the correlated resistor network in the Ising case, at the tricritical percolation point, we performed Monte Carlo calculations at the Ising critical point, and calculated the Ising expectation value of the conductance for different system sizes. We used the Wolff-algorithm [21] for the Monte Carlo part, and the multigrid method of Edwards *et al* [22], based on the standard code AMG1R4 [23], to calculate the conductance of a spin configuration.

To test our program, we calculated the exactly known [7, 8] exponent of the order parameter. The order parameter of percolation is the density of sites, P , in the infinite cluster. From finite size scaling, it follows that this quantity scales with the linear system size L as

$$P(L) \sim L^{-2+y_h} \quad \text{for } L \rightarrow \infty \quad (9)$$

where y_h is the most relevant magnetic eigenvalue. In our Monte Carlo runs, we measured the number of sites in the ‘spanning cluster’, the cluster that extends over the lattice and thus allows for conductance. With the scaling equation, the behaviour of P as a function of the system size yields an estimate of y_h . We calculated P with the system size L running from 7 to 350, and the data are plotted in figure 5. The figure, a log-log plot, shows that the system sizes are too small to exhibit the behaviour of equation (9); corrections to scaling have to be included. We did this, and found $y_h = 1.945 \pm 0.005$ and a correction to scaling

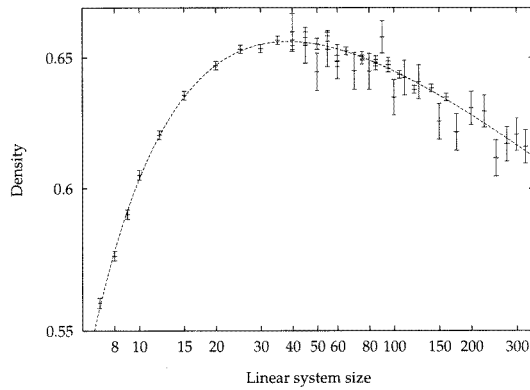


Figure 5. Log–log plot of the density of sites P in the spanning cluster at the Ising critical point, as a function of the linear system size L . The plot shows a far from straight line, meaning that corrections to scaling are important. The broken curve is the result of our fit against the function $P(L) = a_1 L^{-\alpha}(1 + a_2 L^{-\beta})$. The values of α and β are $\alpha = 0.055 \pm 0.005$ and $\beta = 0.92 \pm 0.07$.

exponent $\omega = 0.96 \pm 0.08$. The exact result is $y_h = \frac{187}{96} \approx 1.948$. Our estimate thus agrees well within the error bars.

It is believed that a similar finite size scaling relation is valid for the conductance σ . It should scale with the linear system size, L , as

$$\sigma(L) \sim L^{-t/\nu} \quad (10)$$

where ν is the percolation exponent of the correlation length. We calculated the conductance again with L running from 7 to 350. The data are plotted in figure 6; the log–log plot almost shows a straight line. In fitting the data to equation (10), we tried to include a correction to scaling term, but, due to the almost perfect scaling behaviour, this did not yield sensible results. Therefore we performed the fit against equation (10) without additional terms, yielding the value $t/\nu = 0.2000 \pm 0.0007$. Due to the absence of the correction to scaling term the error in this result might be an underestimation of the actual error.

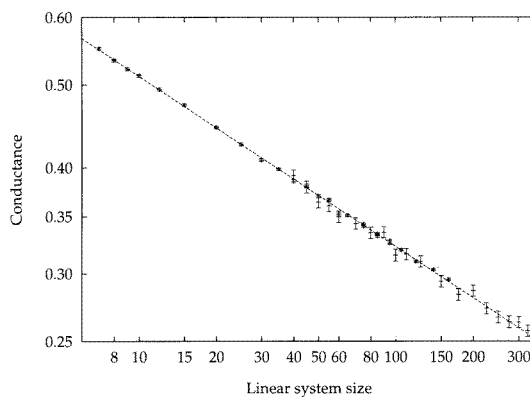


Figure 6. Log–log plot of the conductance σ of the lattice at the Ising critical point, as a function of the linear system size L . The plot shows an almost a straight line. Fitting the scaling behaviour with a correction to scaling term did not yield sensible results. The broken line is the result of our fit against the function $\sigma(L) = aL^{-\alpha}$, giving a value of $\alpha = 0.2000 \pm 0.0007$.

The exponent ν of percolation at the Ising critical point is $\nu = 1$ for the direction parallel to the T -axis and $\nu = \frac{8}{15}$ for the other directions. That means that the exponent t that governs the vanishing conductance at T_c is $t \approx 0.200$ for the temperature direction and $t \approx 0.107$ for the field direction. This is a surprisingly low result, as compared with the t value of the random resistor network, $t \approx 1.30$. The presence of critical correlations thus strongly influences the value of the conductance exponents.

Equation (10) relies on the validity of finite size scaling, and it is in this case not *a priori* clear that it is valid, since there does not exist a rigorously defined renormalization transformation for the conductance, neither for the random resistor network nor for the correlated resistor network. Our case is the latter, and we checked the validity of the scaling assumption by performing calculations for different system sizes and different temperatures around $T = T_c$.

Scaling can be derived if there is a field in the percolation model that couples to the conductance σ and that shows a similar behaviour under renormalization transformations as, e.g. the percolation order parameter. Let us call this field h_c and its corresponding critical exponent y_c . The ‘free energy’ c then obeys

$$c(u_t, h_c, L) = b^{-2}c(b^{y_t}u_t, b^{y_c}h_c, L/b)$$

where b is the renormalization length, u_t is the reduced temperature, and L is the system dimension. Such a field h_c is not known but, assuming that it exists, the scaling relation of the conductance can be derived. The conductance is the derivative with respect to this field:

$$\sigma = \frac{\partial c}{\partial h_c}. \quad (11)$$

Putting $b = L$ and $u_t = h_c = 0$ yields equation (10) with $t/\nu = 2 - y_c$. If we do not set u_t to zero, it follows from equation (11) that $\sigma L^{t/\nu}$ is a function of $L^{y_t}u_t$. Plotting $\sigma L^{t/\nu}$ against $L^{y_t}u_t$ (with $y_t = 1/\nu = 1$ in the Ising model) must display the so-called data collapse: plots for different values of L must collapse onto the same curve.

We performed Monte Carlo calculations around the critical point for different system sizes and plotted the conductance as described above. The plot is shown in figure 7 and

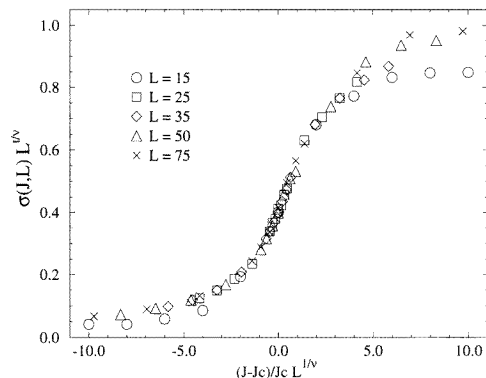


Figure 7. The Ising expectation value of the conductance in the neighbourhood of the critical point for different system sizes L . Error bars have the magnitude of the symbol size. The values for the exponents used are $\nu = 1$ and $t = 0.20$. The data for different system sizes clearly collapse onto the same curve, with small deviations arising only away from the critical point for smaller system sizes. This clearly shows that the concept of scaling applies to the conductance.

clearly shows that the values of the conductance fall onto the same curve. This justifies the validity of the scaling assumption in the case of the correlated resistor network.

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